

# GENERIC REPRESENTATIONS IN $L$ -PACKETS

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**ABSTRACT.** We give the details of the construction of a map to restate a conjectural expression about adjoint group action on generic representations in  $L$ -packets. We give an application of the construction to give another proof of the classification of the Knapp-Stein  $R$ -group associated to a unitary unramified character of a torus. Finally we prove the conjecture for unramified  $L$ -packets.

## 1. INTRODUCTION

Let  $G$  be a quasi-split connected reductive group defined over a local field  $k$  of characteristic zero and let  $Z$  be the center of  $G$ . Let  $B$  be a  $k$ -Borel subgroup of  $G$  and let  $T$  be a maximal  $k$ -torus in  $B$ . Let  $U$  be the unipotent radical of  $B$ . A character  $\psi : U(k) \rightarrow \mathbb{C}^\times$  is called *generic* if the stabilizer of  $\psi$  in  $T(k)$  is exactly the center  $Z(k)$ . An irreducible admissible representation  $\pi$  of  $G$  is called *generic* ( $\psi$ -*generic*) if there exists a generic character  $\psi$  of  $U(k)$  such that  $\text{Hom}_{G(k)}(\pi, \text{Ind}_{U(k)}^{G(k)} \psi) \neq 0$ .

The conjectural *local Langlands program* partitions the irreducible admissible representations of  $G$  into finite sets known as  $L$ -packets. Each  $L$ -packet is expected to be parametrized by an arithmetic object called the *Langlands parameter*, which is an *admissible homomorphism* from the *Weil-Deligne* group  $W'_k$  of  $k$  to the  $L$ -group  ${}^L G$  of  $G$ . See [Bor79a] for the definitions and statements.

To each Langlands parameter  $\varphi$ , one can associate a finite group  $\mathcal{S}_\varphi$  (see [Art06, Section 1, eq. (1.1)]). It is expected that the associated  $L$ -packet  $\Pi_\varphi$  is parametrized by the irreducible representations  $\widehat{\mathcal{S}_\varphi}$  of  $\mathcal{S}_\varphi$  [Art06, Section 1]. The parametrization will depend on the choice of a Whittaker datum for  $G$ , which is a  $G(k)$ -conjugacy class of pairs  $(B, \psi)$ , where  $\psi$  is a generic character of  $U(k)$ . When  $\Pi_\varphi$  is generic, i.e., it has a generic representation, the  $\psi$ -generic representation in  $\Pi_\varphi$  is then required to correspond to the trivial representation of  $\mathcal{S}_\varphi$ . The parametrization is also expected to satisfy certain conjectural endoscopic character identity [Kal13].

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When  $\varphi$  is a *tempered* parameter, i.e., a parameter whose image projects onto a relatively compact subset of the complex dual  $\hat{G}$  of  $G$ , Shahidi's *tempered  $L$ -packet conjecture* [Sha90, §9] states that  $\Pi_\varphi$  must be generic.

Let  $\Gamma_k$  be the absolute Galois group of  $k$  and write  $H^1(k, -)$  for  $H^1(\Gamma_k, -)$ . In Section 3, we construct a map  $\gamma_\varphi : R_\varphi := \pi_0(Z_{\hat{G}}(\text{Im}(\varphi))) \rightarrow H^1(k, X(Z))$ , where  $X(Z)$  is the character lattice of  $Z$  and  $\varphi$  is any Langlands parameter. Using Tate duality, we get the dual map  $\hat{\gamma}_\varphi : H^1(k, Z) \rightarrow \widehat{R_\varphi}$ , where  $\widehat{R_\varphi}$  is the set of irreducible representations of  $R_\varphi$ . Let  $p : t \in T \rightarrow \bar{t} \in T_{\text{ad}} := T/Z$  be the adjoint morphism. The finite abelian group  $T_{\text{ad}}(k)/p(T(k)) \hookrightarrow H^1(k, Z)$  acts simply transitively on the set of  $T(k)$ -orbits of generic characters [DR10, §3]. The map  $\zeta_\varphi := \hat{\gamma}_\varphi|_{T_{\text{ad}}(k)/p(T(k))}$  factors through  $\widehat{\mathcal{S}_\varphi}$  (see [GGP12, Sec. 9(4)], also [Kal13, Sec. 3]).

Now fix a parametrization  $\rho \in \widehat{\mathcal{S}_\varphi} \mapsto \pi_\rho \in \Pi_\varphi$  by making the choice of a Whittaker datum. The following is a version of the conjecture in [GGP12, Sec. 9(3)] for generic  $L$ -packets.

**Conjecture.** *A representation  $\pi_\rho \in \Pi_\varphi$  is  $\psi$ -generic iff  $\pi_{t \cdot \rho}$  is  $t \cdot \psi$  generic for all  $t \in T_{\text{ad}}(k)$ , where  $t \cdot \rho := \rho \otimes \zeta_\varphi(t)$ .*

The map  $\hat{\gamma}_\varphi$  was constructed in [Kuo10] in a very special case ( $G$  split semisimple and  $\varphi$  is the parameter associated to a unitary character of  $T(k)$ ). For depth zero supercuspidal  $L$ -packets, the conjecture follows from [DR10]. When  $G$  is semisimple and split and the  $L$ -packet is formed by the constituents of a unitary principal series, the conjecture follows from [Kuo02]. In [Kal13], Kaletha gives a proof of the above conjecture for classical groups using very general arguments.

Now let  $G$  be unramified and let  $\varphi$  be the parameter associated to a unitary unramified character  $\lambda$  of  $T(k)$ . The construction of the map  $\gamma_\varphi$  allows one to obtain a nice description of the group  $R_\varphi$  as a certain subgroup of an extended affine Weyl group (Proposition 10). Using this, in Theorem 11, we obtain in a conceptual and uniform way, the classification of the Knapp-Stein  $R$ -group associated to  $\lambda$ . This kind of classification was obtained by Keys [Key82, §3] in a case by case manner. For split groups, using different methods, another way of getting the classification obtained by Keys was recently given by Kamran and Plymen [KP13]. Our situation is more general and we also describe the isomorphism, which has a simple description.

Finally in Theorem 12, we prove the conjecture for *unramified  $L$ -packets* (see Sec. 5). We do not assume the packet to be tempered.

## 2. PRELIMINARIES

**2.1. Group Cohomology.** For details about this subsection, see [Ser97, Ch. 5].

Let  $\Gamma$  be a topological group and let

$$(2.1) \quad 1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

be a short exact sequence of  $\Gamma$ -groups. Assume that  $A$  is central subgroup of  $B$ . Then  $C$  acts on  $B$  by inner automorphisms and it acts trivially on  $A$ . Let  $\gamma : \Gamma \rightarrow C$  be a co-cycle in  $C$ , i.e., it satisfies the relation  $\gamma(ab) = \gamma(a)^a \gamma(b)$  for all  $a, b \in \Gamma$ . By twisting the short exact sequence in (2.1) by  $\gamma$ , we get another short exact sequence

$$1 \rightarrow A \rightarrow {}_\gamma B \rightarrow {}_\gamma C \rightarrow 1$$

From this we get a long exact cohomology sequence

$$\begin{aligned} 1 &\rightarrow H^0(\Gamma, A) \rightarrow H^0(\Gamma, {}_\gamma B) \rightarrow H^0(\Gamma, {}_\gamma C) \rightarrow H^1(\Gamma, A) \rightarrow \\ &\rightarrow H^1(\Gamma, {}_\gamma B) \rightarrow H^1(\Gamma, {}_\gamma C). \end{aligned}$$

## 2.2. Affine roots and affine transformations.

2.2.1. *The group  $\Omega$ .* Let  $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$  be a based root datum in the sense of [Spr79, 1.9]. So  $X$  and  $\check{X}$  are free abelian groups in duality by a pairing  $X \times \check{X} \rightarrow \mathbb{Z}$ ,  $R$  is a root system in the vector space  $Q \otimes \mathbb{R}$ , where  $Q$  is the root lattice,  $\check{R}$  is the set of co-roots,  $\Delta \subset R$  is a basis and  $\check{\Delta}$  is the dual basis. Let  $W = W(\Psi)$  be the Weyl group. The set  $\Delta$  determines an alcove  $C$  in  $V := X \otimes \mathbb{R}$  in the following way. Let  $\check{\Delta} = \{\check{\alpha}_1, \dots, \check{\alpha}_l\}$  and let  $\check{\beta} = \sum_{i=1}^l n_i \check{\alpha}_i$  be the highest co-root. Then  $C$  is the alcove in  $V$  defined by  $C = \{x \in V : \check{\alpha}_0(x) \geq 0, \dots, \check{\alpha}_l(x) \geq 0\}$ , where  $\check{\alpha}_0 = 1 - \check{\beta}$ . Let  $\tilde{W} = W \ltimes X$  and  $\tilde{W}^\circ = W \ltimes Q$ . Let  $\Omega$  be the stabilizer of  $C$  in  $\tilde{W}$ . Then  $\tilde{W} = \Omega \ltimes \tilde{W}^\circ$ .

Now assume that  $\Psi$  is *semisimple* [Spr79, 1.1] and  $R$  is an irreducible root system in  $V$ . Let  $c_0$  be the *weighted barycenter* of  $C$ , characterized by the equations  $\check{\alpha}_i(c_0) = 1/h$  for  $i = 0, \dots, l$ , where  $h$  is the Coxeter number. For any  $w \in W$ , let  $\tilde{w}$  be the affine map  $x \in V \mapsto w(x - c_0) + c_0$ . It is the unique affine map fixing  $c_0$  with tangent part  $w$ . The following lemmas follow from [AYY13, Lemma 6.2].

**Lemma 1.** *For any  $w \in W$ , the following are equivalent:*

- (1)  $\tilde{w} \in \tilde{W}$ .
- (2)  $\tilde{w} \in \Omega$ .

**Lemma 2.** *There is an isomorphism  $\iota : \Omega \rightarrow X/Q$  defined by any of the following ways*

- (1)  $\iota(\tilde{w}) = (w^{-1} - 1)c_0 + Q$ .
- (2) The natural projection  $\tilde{W} \rightarrow \tilde{W}/\tilde{W}^\circ = X/Q$  restricted to  $\Omega$ .

2.2.2. *Based root datum.* Let  $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$  be a reduced based root datum. Let  $\theta$  be a finite group acting on  $\Psi$ . In [Yu], Jiu-Kang Yu defines the following 6-tuple  $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$ :

$$\begin{aligned} \underline{X} &= X_\theta / \text{torsion}, \\ \underline{\check{X}} &= \check{X}^\theta, \\ \underline{R} &= \{\underline{a} : a \in R\}, \quad \text{where } \underline{a} := a|_{\underline{\check{X}}} \\ \underline{\check{R}} &= \{\check{\alpha} : \alpha \in \underline{R}\}, \\ \underline{\Delta} &= \{\underline{a} : a \in \Delta\}, \\ \underline{\check{\Delta}} &= \{\check{\alpha} : \alpha \in \underline{\Delta}\}. \end{aligned}$$

The explanation for the defining formulas is as follows. We first note that  $\underline{X}$  and  $\underline{\check{X}}$  are free abelian groups, dual to each other under the canonical pairing  $(\underline{x}, y) \mapsto \langle x, y \rangle$ , for  $\underline{x} \in \underline{X}$ ,  $y \in \underline{\check{X}} \subset \check{X}$ , where  $x$  is any preimage of  $\underline{x}$  in  $X$ . Define  $\check{\alpha}$  for  $\alpha \in \underline{R}$  as follows:

$$(2.2) \quad \check{\alpha} = \begin{cases} \sum_{a \in R: a|_{\underline{\check{X}}} = \alpha} \check{a}, & \text{if } 2\alpha \notin \underline{R} \\ 2 \sum_{a \in R: a|_{\underline{\check{X}}} = \alpha} \check{a}, & \text{if } 2\alpha \in \underline{R} \end{cases}$$

In [Yu], Jiu-Kang Yu proves the following.

**Theorem 3.** [Jiu-Kang Yu] *The 6-tuple  $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$ , with the canonical pairing between  $\underline{X}$  and  $\underline{\check{X}}$  and the correspondence  $\underline{R} \rightarrow \underline{\check{R}}, \alpha \mapsto \check{\alpha}$ , is a based root datum. Moreover, the homomorphism  $W(\Psi)^\sigma \rightarrow \mathbf{GL}(\underline{\check{X}})$ ,  $w \mapsto w|_{\underline{\check{X}}}$  is injective and the image is  $W(\underline{\Psi})$ .*

The above Theorem for simply connected groups is proved in [Ree10, Sec. 3.3].

### 3. A CONSTRUCTION AND A CONJECTURE

3.1. **Construction.** Let  $G$  be a quasi-split group defined over a local field  $k$  of characteristic zero. Let  $T$  be a maximal  $k$ -torus of  $G$  which is contained in a  $k$ -Borel subgroup  $B$ . Let  $\hat{G}_{\text{sc}}$  be the simply connected cover of the derived group  $\hat{G}_{\text{der}}$  of  $\hat{G}$ , where  $\hat{G}$  is the complex dual of  $G$ . Let  $\hat{T} \subset \hat{G}$  be the torus dual to  $T$  and  $\hat{T}_{\text{sc}}$  be the pull back of  $(\hat{T} \cap \hat{G}_{\text{der}})^\circ$  via  $\hat{G}_{\text{sc}} \rightarrow \hat{G}_{\text{der}}$ . Let  $X = X(T)$  (resp.  $\check{X} = \check{X}(T)$ ) denote the group of characters (resp. co-characters) of  $T$ . Let  $Z$  be the center of  $G$  and let  $\hat{\mathfrak{z}}$  be the Lie algebra of the center  $\hat{Z}$  of  $\hat{G}$ . Then  $\tilde{G} := \hat{G}_{\text{sc}} \times \hat{\mathfrak{z}}$  is the topological universal cover of  $\hat{G}$ . We have a short exact sequence

$$(3.1) \quad 1 \rightarrow \pi_1(\hat{G}) \rightarrow \tilde{G} \rightarrow \hat{G} \rightarrow 1,$$

where  $\pi_1(\hat{G})$  is the topological fundamental group of  $\hat{G}$ . Let  $Q$  denote the root lattice. Then from [Spr79, 2.15],

$$(3.2) \quad X(Z) \cong X/Q.$$

The algebraic fundamental group of  $\hat{G}$  is  $\check{X}(\hat{T})/\check{X}(\hat{T}_{\text{sc}}) = X/Q$ . Since  $\hat{G}$  is a complex algebraic group, its algebraic fundamental group is the same as its topological fundamental group (see [BGA14]). Therefore

$$(3.3) \quad X/Q \cong \pi_1(\hat{G}).$$

Let  $W_k$  (resp.  $\Gamma_k$ ) denote the Weil group (resp. absolute Galois group) of  $k$ . Define  $W'_k := W_k$  if  $k$  is archimedean and  $W'_k := W_k \times \text{SL}(2, \mathbb{C})$  if  $k$  is non-archimedean.  $W'_k$  is called the Weil-Deligne group of  $k$ . Let  $\varphi : W'_k \rightarrow {}^L G$  be a Langlands parameter (see [Bor79b, Sec. 8.2]). View  $\varphi$  as an admissible homomorphism. Then  $\varphi$  determines a co-cycle  $\phi|_{W_k} : W_k \rightarrow {}^L G \rightarrow \hat{G}$ . We can twist the exact sequence (3.1) by the co-cycle  $\phi$  (see Section 2.1). Then using the isomorphism  $X(Z) \cong \pi_1(\hat{G})$ , we get

$$\tilde{\gamma} : H^0(W_k, \phi \hat{G}) \rightarrow H^1(W_k, X(Z)).$$

Since  $H^0(W_k, \phi \hat{G}) \supset Z_{\hat{G}}(\text{Im}(\varphi))$ , by restriction this induces

$$\tilde{\gamma}' : Z_{\hat{G}}(\text{Im}(\varphi)) \rightarrow H^1(W_k, X(Z)).$$

Since this map is continuous and  $H^1(W_k, X(Z))$  is discrete,  $\ker(\tilde{\gamma}') \supset (Z_{\hat{G}}(\text{Im}(\varphi)))^\circ$ . Thus we get a map

$$(3.4) \quad \gamma'_\varphi : R_\varphi := \pi_0(Z_{\hat{G}}(\text{Im}(\varphi))) \rightarrow H^1(W_k, X(Z)).$$

Since  $R_\varphi$  is finite,  $\gamma'_\varphi$  induces

$$\gamma''_\varphi : R_\varphi \rightarrow H^1(W_k, X(Z))^{\text{tor}}.$$

By [Kar11, Theorem 4.1.3 (ii)], we have a functorial isomorphism

$$H^1(W_k, X(Z))^{\text{tor}} = H^1(k, X(Z)).$$

Here we are abbreviating  $H^1(\Gamma_k, -)$  by the notation  $H^1(k, -)$ . We thus get a map

$$(3.5) \quad \gamma_\varphi : R_\varphi \rightarrow H^1(k, X(Z)).$$

By Tate Duality ([Mil06, Corr. 2.4]), we have an isomorphism

$$(3.6) \quad H^1(k, X(Z)) \cong \text{Hom}(H^1(k, Z), \mathbb{C}^\times).$$

Using the isomorphism (3.6) in (3.5), we get a map

$$(3.7) \quad \hat{\gamma}_\varphi : H^1(k, Z) \rightarrow \widehat{R_\varphi},$$

where  $\widehat{R_\varphi}$  is the set of irreducible representations of  $R_\varphi$ . Since  $H^1(k, X(Z))$  is abelian, the image of  $\hat{\gamma}_\varphi$  lies in the group of one dimensional representations of  $R_\varphi$ .

**3.2. Statement of a conjecture.** Let  $U$  be the unipotent radical of  $B$  and let  $p : G \rightarrow G_{\text{ad}} := G/Z$  be the adjoint morphism. We denote by the same symbol, the induced map  $p : T \rightarrow T_{\text{ad}} := T/Z$ .

**Definition 4.** A character  $\psi : U(k) \rightarrow \mathbb{C}^\times$  is *generic* if its stabilizer in  $T_{\text{ad}}(k)$  is trivial.

The group  $T_{\text{ad}}(k)$  acts simply transitively on the set of generic characters of  $U(k)$ . Hence the finite abelian group  $T_{\text{ad}}(k)/p(T(k))$  acts simply transitively on the set of  $T(k)$ -orbits of generic characters.

**Definition 5.** The *pure inner forms* of  $G$  are the groups  $G'$  over  $k$  which are obtained by inner twisting by elements in the pointed set  $H^1(k, G)$ .

All pure inner forms have the same center  $Z$  over  $k$ . Let  $G'$  be a pure inner form of  $G$ . Denote the maximal torus of  $G'$  (resp.  $G'_{\text{ad}}$ ) corresponding to  $T$  (resp.  $T_{\text{ad}}$ ) by  $T'$  (resp.  $T'_{\text{ad}}$ ). We will denote the adjoint morphism for all inner forms by the same symbol  $p$ .

We have a canonical inclusion  $T'_{\text{ad}}(k)/p(T'(k)) \hookrightarrow H^1(k, Z)$  and a canonical isomorphism  $T'_{\text{ad}}(k)/p(T(k)) \cong G'_{\text{ad}}(k)/p(G'(k))$  (Lemma 5.1 [DR10]). Equation (3.7) thus induces

$$\zeta'_\varphi : G'_{\text{ad}}(k)/p(G'(k)) \rightarrow \widehat{R_\varphi}.$$

Let  $\tilde{\Pi}_\varphi$  denote the *Vogan  $L$ -packet* associate to  $\varphi$ . It is the union of the standard  $L$ -packets associated to  $\varphi$  of  $G$  and all its pure inner forms. By standard, we mean  $L$ -packets as defined in [Bor79a]. Let  $\rho \in \widehat{R_\varphi} \mapsto \pi_\rho \in \tilde{\Pi}_\varphi$  be the parametrization defined after the choice of a Whittaker datum. Assume that this parametrization is compatible with Deligne's normalization of the local Artin map (see [GGP12, Sec. 3]). Let  $\Pi'_\varphi$  be the standard  $L$ -packet of  $G'$  contained in  $\tilde{\Pi}_\varphi$ . The following is a conjecture in [GGP12, Sec. 9 (3)].

**Conjecture 6.** For  $g \in G'_{\text{ad}}(k)$ ,  $\pi_\rho \circ \text{Ad}(g) = \pi_{g \cdot \rho}$ , where  $g \cdot \rho = \rho \otimes \zeta'_\varphi(g)$  and  $\pi_\rho \in \Pi'_\varphi$ . Thus  $\pi_\rho$  is  $\psi$ -generic iff  $\pi_{g \cdot \rho}$  is  $g \cdot \psi$  generic.

We have a natural inclusion  $\pi_0(\hat{Z}^{\Gamma_k}) \subset \widehat{R_\varphi}$ . Let  $\tau \in \widehat{R_\varphi}$ . In [GGP12, Sec. 9(4)], it is explained that the pure inner form of  $G$  which acts on the representation corresponding to the parameter  $(\varphi, \tau)$  is determined by the character  $\tau|_{\pi_0(\hat{Z}^{\Gamma_k})}$ . Thus the standard  $L$ -packet  $\Pi_\varphi \subset \tilde{\Pi}_\varphi$  of  $G$  is parametrized by  $\tau \in \widehat{R_\varphi}$  whose restriction to  $\pi_0(\hat{Z}^{\Gamma_k})$  is trivial. In other words, the standard  $L$ -packet is parametrized by the irreducible representations  $\widehat{\mathcal{S}_\varphi} \hookrightarrow \widehat{R_\varphi}$  of the group  $\mathcal{S}_\varphi := \pi_0(Z_{\hat{G}}(\text{Im}(\varphi))/\hat{Z}^{\Gamma_k})$ . The map  $\zeta_\varphi : G_{\text{ad}}(k)/p(G(k)) \rightarrow \widehat{R_\varphi}$  thus must factor through  $\widehat{\mathcal{S}_\varphi}$ . Conjecture 6 for standard generic  $L$ -packets can be stated as:

**Conjecture 6'.**  $\pi_\rho \in \Pi_\varphi$  is  $\psi$ -generic iff  $\pi_{g \cdot \rho}$  is  $g \cdot \psi$  generic, where  $\rho \in \widehat{R}_\varphi$ ,  $g \in G_{\text{ad}}(k)$ , and where  $g \cdot \rho = \rho \otimes \zeta_\varphi(g)$ .

*Remark 7.* In [Kal13, Sec. 3], Kaletha constructs a map  $\zeta_\varphi : G_{\text{ad}}(k)/p(G(k)) \rightarrow \widehat{\mathcal{S}}_\varphi$ . In [Kal13, Sec. 1, eq. (1.1)], he states the above conjecture in a more precise manner by comparing the parametrization of a tempered  $L$ -packet for different choices of Whittaker data. He also points out that the action of  $g \in G_{\text{ad}}(k)$ , should send  $\rho \in \widehat{\mathcal{S}}_\varphi$  to  $\rho \otimes \zeta_\varphi(g)$  or  $\rho \otimes \zeta_\varphi^{-1}(g)$  depending on which of the two possible normalizations of the local Artin map one chooses. The normalization in Conjecture 6 uses Deligne's normalization [GGP12, Sec. 3].

#### 4. DESCRIPTION OF $R$ -GROUP

Let the notations be as in Section 3. Assume that  $G$  is unramified, i.e., it is quasi-split and split over an unramified extension of  $k$ . We also assume  $k$  to be non-archimedean. Let  $I$  be the inertia subgroup of  $W_k$  and let  $\sigma$  be the Frobenius element in  $W_k/I$ . Throughout this section, we will abbreviate  $H^1(W_k/I, -)$  by the notation  $H^1(\sigma, -)$ .

**4.1. Case of an unramified parameter.** Let  $\bar{s} \in \hat{T}$  and let  $\varphi$  be the Langlands parameter determined by the map  $\sigma \mapsto \bar{s}$ . Let  $s$  be a lift of  $\bar{s}$  in  $\hat{T}_{\text{sc}} \times \hat{\mathfrak{z}}$ .

Let  $H^1(\sigma, \tilde{G})_{\text{ss}} \subset H^1(\sigma, \tilde{G})$  denote the  $\sigma$ -conjugacy classes of the semisimple elements of  $\tilde{G}$ , where  $\tilde{G} = \hat{G}_{\text{sc}} \times \hat{\mathfrak{z}}$  as in Section 3. Denote by  $[t]$ , the class of  $t \in \tilde{G}_{\text{ss}}$  in  $H^1(\sigma, \tilde{G})_{\text{ss}}$ . Let  $A := \pi_1(\hat{G})$ . Let  $\underline{A}$  denote  $A_\sigma$ , the co-invariant of  $A$  with respect to  $\sigma$ . We have  $H^1(\sigma, A) \cong \underline{A}$ . Let  $\underline{x}$  denote the image of  $x \in A$  in  $\underline{A}$ . Then there is an action of  $H^1(\sigma, A)$  on  $H^1(\sigma, \tilde{G})_{\text{ss}}$  given by

$$\underline{x} \cdot [t] := [xt] \quad \text{for } x \in A, t \in \tilde{G}_{\text{ss}}.$$

Denote by  $\underline{A}_\varphi$  the stabilizer of  $[s]$  in  $\underline{A}$ .

**Lemma 8.** *The map  $\gamma_\varphi$  in equation (3.5) induces an isomorphism  $R_\varphi \cong \underline{A}_\varphi$ .*

*Proof.* We have

$$\begin{aligned} R_\varphi &\cong \ker(H^1(\sigma, A) \rightarrow H^1(\sigma, \tilde{G})) \\ &= \{ \underline{x} \in \underline{A} \mid g^{-1}xs(\sigma g)s^{-1} = 1 \quad \text{for some } g \in \tilde{G} \} \\ &= \{ \underline{x} \in \underline{A} \mid \underline{x} \cdot [s] = [s] \} \\ &= \underline{A}_\varphi. \end{aligned}$$

□

*Remark.* The above Lemma is also proved in [Yu].

**4.2. Action of  $\Omega$ .** Let  $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$  be the based root datum of  $(G, B, T)$ . So  $X$  (resp.  $\check{X}$ ) is the group of characters (resp. co-characters) of  $T$ ,  $R$  (resp.  $\check{R}$ ) is the set of roots (resp. co-roots) of  $T$  in the Lie algebra of  $G$  and  $\Delta$  (resp.  $\check{\Delta}$ ) is a basis in  $R$  (resp.  $\check{R}$ ) determined by  $B$ . Let  $\underline{\Psi} = (\underline{X}, \underline{R}, \underline{\Delta}, \underline{\check{X}}, \underline{\check{R}}, \underline{\check{\Delta}})$  be the based root datum obtained from  $\Psi = (X, R, \Delta, \check{X}, \check{R}, \check{\Delta})$  by the construction given in 2.2.2. Let  $\underline{Q}$  be the lattice generated by  $\underline{R}$ . Let  $\underline{C}$  be the alcove in  $\underline{V} := \underline{X} \otimes \mathbb{R}$  determined by  $\underline{\Delta}$ . Let  $W = W(\underline{\Psi})$  be the Weyl group of the based root datum  $\underline{\Psi}$ . By Theorem 3, it is the relative Weyl group of  $G$ . Let  $\underline{\Omega} \cong \underline{X}/\underline{Q}$  be the stabilizer in  $W \ltimes \underline{X}$  of  $\underline{C}$  (see Section 2.2.1).

By [Bor79a, Lemma 6.5] (or more directly by [Mis15, Prop. 11]), we have

$$(4.1) \quad \hat{T}_\sigma/W \cong (\hat{G} \rtimes \sigma)_{\text{ss}}/\text{Int}(\hat{G}),$$

where  $(\hat{G} \rtimes \sigma)_{\text{ss}}$  is the set of semisimple elements in  $\hat{G} \rtimes \sigma$  and  $\text{Int}(\hat{G})$  denotes the group of inner automorphisms of  $\hat{G}$ .

Let  $\hat{T}^{\text{cpt}}$  be the maximal compact subtorus in  $\hat{T}$ . Write  $\hat{T} = X \otimes \mathbb{C}^\times$ . Under this identification,  $\hat{T}^{\text{cpt}} = X \otimes (\mathbb{R}/\mathbb{Z}) \cong X \otimes \mathbb{R}/X$ . Let  $\hat{G}^{\text{cpt}}$  be the set of those semi-simple elements of  $\hat{G}$  which lie in some maximal compact subtorus of  $\hat{G}$ . The isomorphism in (4.1) induces an isomorphism

$$(4.2) \quad \begin{aligned} \hat{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\hat{G}) &\cong (\hat{T}^{\text{cpt}})_\sigma / W \\ &\cong \underline{X} \otimes \mathbb{R} / W \ltimes \underline{X} \\ &= \underline{X} \otimes \mathbb{R} / ((W \ltimes \underline{Q}) \rtimes \underline{\Omega}) \end{aligned}$$

$$(4.3) \quad \longleftrightarrow \bar{\underline{C}} / \underline{\Omega},$$

where  $\bar{\underline{C}}$  is the closure of the alcove  $\underline{C}$  determined by  $\underline{\Delta}$ .

Let  $\hat{\mathfrak{z}}^{\text{cpt}} := X/Q \otimes \mathbb{R}$ . It is the Lie algebra of the maximal compact subtorus of  $\hat{Z}$ . Let  $\tilde{G}^{\text{cpt}} = \hat{G}_{\text{sc}}^{\text{cpt}} \times \hat{\mathfrak{z}}^{\text{cpt}}$ . Then

$$(4.4) \quad \begin{aligned} \tilde{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\tilde{G}) &\cong \tilde{T}_\sigma^{\text{cpt}} / W \\ &\cong ((\underline{X}_{\text{sc}} \otimes (\mathbb{R}/\mathbb{Z})) \times (\underline{X}/\underline{Q} \otimes \mathbb{R})) / W \\ &\cong \underline{X} \otimes R / (\underline{Q} \rtimes W) \quad \text{since } \underline{X}_{\text{sc}} = \underline{Q} \end{aligned}$$

$$(4.5) \quad \longleftrightarrow \bar{\underline{C}}.$$

We have  $\underline{A} \cong (X/Q)_\sigma \rightarrow \underline{X}/\underline{Q} \cong \underline{\Omega}$ . In Lemma 9 below, we will show that the action of  $\underline{A}$  on  $\tilde{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\tilde{G}) \subset (\tilde{G} \rtimes \sigma)_{\text{ss}} / \text{Int}(\tilde{G})$  is compatible with the action of  $\underline{\Omega}$  on  $\bar{\underline{C}}$ . Now  $G$  is isogenous to  $Z^\circ \times (G_{\text{sc}})_{\text{der}}$ , where  $(G_{\text{sc}})_{\text{der}}$  is the simply connected cover of the derived group of  $G$  and  $Z^\circ$  is the identity component of the center of  $G$ . Since any simply connected semisimple group is the direct product of



almost simple groups, it suffices to prove the compatibility in the case when  $G$  is almost simple.

Let  $\underline{a} \in \underline{A}$  and let  $a$  be a lift of  $\underline{a}$  in  $A$ . Let  $\underline{c}_0$  be the weighted barycenter of  $\underline{C}$  and let  $\underline{a} \mapsto \tilde{\omega}_a$  under the surjection  $\underline{A} \twoheadrightarrow \underline{\Omega}$ , where  $\omega_a \in W$  and  $\tilde{\omega}_a$  is the affine transformation  $x \in \underline{X} \otimes \mathbb{R} \mapsto \omega_a(x - \underline{c}_0) + \underline{c}_0$  (see Section 2.2.1).

Let  $[s] \mapsto x_{[s]}$  under the bijection  $\tilde{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\tilde{G}) \leftrightarrow \underline{\bar{C}}$ , where  $[s]$  denotes the class of  $s \in \tilde{G}^{\text{cpt}}$ . Without loss of generality, we can assume that  $s \in \tilde{T}^{\text{cpt}}$ .

**Lemma 9.** *We have  $\tilde{\omega}_a \cdot x_{[s]} = x_{[as]}$ .*

*Proof.* Let  $\tilde{W}^\circ = W \ltimes \underline{Q}$ . We have

$$\begin{aligned} \tilde{\omega}_a \cdot x_{[s]} &= \omega_a(x_{[s]} - \underline{c}_0) + \underline{c}_0 \\ &= \omega_a \cdot x_{[s]} + (1 - \omega_a)\underline{c}_0 \\ &= \omega_a(x_{[s]} + (\omega_a^{-1} - 1)\underline{c}_0). \end{aligned}$$

By Lemma 2,  $\tilde{\omega}_a \mapsto (\omega_a^{-1} - 1)\underline{c}_0 + \underline{Q}$  under the isomorphism  $\underline{\Omega} \cong \underline{X}/\underline{Q}$ . Using this we get that  $x_{[a]} \equiv (\omega_a^{-1} - 1)\underline{c}_0 \pmod{\tilde{W}^\circ}$ . Thus

$$\begin{aligned} \tilde{\omega}_a \cdot x_{[s]} &\equiv x_{[s]} + x_{[a]} \pmod{\tilde{W}^\circ} \\ &\equiv x_{[as]} \pmod{\tilde{W}^\circ}. \end{aligned}$$

Since  $\tilde{\omega}_a \cdot x_{[s]} \in \underline{\bar{C}}$  and  $x_{[as]} \in \underline{\bar{C}}$ , we conclude that

$$\tilde{\omega}_a \cdot x_{[s]} = x_{[as]}.$$

□

**4.3. Tempered parameter.** Let  $\lambda$  be a unitary unramified character of  $T(k)$ . Let  $\lambda \mapsto [\bar{s}]$  under the bijection

$$\text{Hom}(T(k), \mathbb{S}^1)/W \cong \hat{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\hat{G}),$$

where  $\bar{s}$  can be chosen to be in  $\hat{T}$ . Here  $\mathbb{S}^1$  denotes the unit circle in  $\mathbb{C}$ . Let  $\varphi$  be the Langlands parameter determined by the map  $\sigma \mapsto \bar{s}$ . Let  $s$  be a lift of  $\bar{s}$  in  $\hat{T}_{\text{sc}} \times \hat{\mathfrak{z}}$ .

Let  $\underline{\Omega}_\varphi$  be the stabilizer of  $x_{[s]} \in \underline{\bar{C}}$  in  $\underline{\Omega}$ . We have

**Proposition 10.**  $\mathcal{S}_\varphi \cong \underline{\Omega}_\varphi$ .

*Proof.* By [Key87, Lem. 2.5(iii)],  $\pi_0(\hat{Z}^\sigma) = \pi_0(\hat{T}^\sigma)$ . But  $\pi_0(\hat{T}^\sigma) \cong (X_\sigma)^{\text{tor}}$ . Since  $\mathcal{S}_\varphi \cong R_\varphi / \pi_0(\hat{Z}^\sigma)$ , by Lemma 8 we get that  $\mathcal{S}_\varphi \cong \underline{A}_\varphi / (X_\sigma)^{\text{tor}} \cong \underline{\Omega}_\varphi$ . Lemma 9 shows that  $\mathcal{S}_\varphi$  and  $\underline{\Omega}_\varphi$  have compatible actions on  $\tilde{G}^{\text{cpt}} \rtimes \sigma / \text{Int}(\tilde{G})$  and  $\underline{\bar{C}}$  respectively. □

When  $G$  is almost simple and simply connected, the non-trivial  $\underline{\Omega}$  are given by the table below (see [Kan01, Sec. 9-4] and [Ree10, Table-1]).

	$\underline{\Omega}$
$A_n$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$B_n$	$\mathbb{Z}/2\mathbb{Z}$
$C_n$	$\mathbb{Z}/2\mathbb{Z}$
$D_n$ ( $n$ even)	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$D_n$ ( $n$ odd)	$\mathbb{Z}/2\mathbb{Z}$
$E_6$	$\mathbb{Z}/3\mathbb{Z}$
$E_7$	$\mathbb{Z}/2\mathbb{Z}$
${}^2A_{2n-1}$ ( $n \geq 3$ )	$\mathbb{Z}/2\mathbb{Z}$
${}^2D_{n+1}$ ( $n \geq 2$ )	$\mathbb{Z}/2\mathbb{Z}$

TABLE 1.

Let  $R_\lambda$  be the Knapp-Stein  $R$ -group associated to  $\lambda$  (see [Key87, §2] for definition). By [Key87, Prop. 2.6],  $R_\lambda \cong \mathcal{S}_\varphi$ . Using Proposition 10, we obtain  $\underline{\Omega}_\varphi \cong R_\lambda$ . In fact, the isomorphism is given by the restriction of the natural projection  $W \ltimes \underline{X} \rightarrow W$  to  $\underline{\Omega}_\varphi$ . We get

**Theorem 11.** *Let  $G$  be an almost simple, simply connected, unramified group defined over a non-archimedean local field  $k$ . The non-trivial  $R_\lambda$  that can appear are precisely the subgroups of  $\underline{\Omega}$  in table 1.*

(see also [KP13]).

This gives the classification obtained by Keys in [Key82, §3] in the case of unramified groups.

## 5. UNRAMIFIED $L$ -PACKET

Let the notations be as in Section 3. Assume further that  $G$  is unramified and that  $k$  is non-archimedean. As in Section 4, let  $I$  be the inertia subgroup of  $W_k$  and let  $\sigma$  be the Frobenius element in  $W_k/I$ .

An *unramified  $L$ -packet* consists of those irreducible subquotients of an unramified principal series representation of  $G(k)$  which have a non-zero vector fixed by some hyperspecial subgroup of  $G(k)$ . Unramified  $L$ -packets are in bijective correspondence with  $(\hat{G} \rtimes \sigma)_{\text{ss}}/\text{Int}(\hat{G})$ . Let  $\varphi$  be a Langlands parameter determined by the  $\sigma$ -conjugacy class of a semi-simple element and let  $\Pi_\varphi$  be the associated unramified  $L$ -packet. The  $L$ -packet  $\Pi_\varphi$  is parametrized by  $\widehat{\mathcal{S}}_\varphi$ , where  $\mathcal{S}_\varphi := \pi_0(Z_{\hat{G}}(\text{Im}(\varphi))/\hat{Z}^{\Gamma_k})$  as in Section 3, after making the choice of a hyperspecial point. We denote the parametrization by  $\rho \in \widehat{\mathcal{S}}_\varphi \mapsto \pi_\rho \in \Pi_\varphi$ .

Let  $K$  be a compact subgroup of  $G(k)$ . Denote by  $[K]$ , the  $G(k)$ -conjugacy class of  $K$ . If  $\pi$  is a representation of  $G(k)$ , we denote by  $\pi^K$  the  $K$ -fixed points of the space realizing  $\pi$ . By the notation  $\pi^{[K]} \neq 0$ , we mean that  $\pi$  has a non-zero vector fixed by some (therefore any) conjugate of  $K$ .

The conjugacy classes of hyperspecial subgroups of  $G(k)$  form a single orbit under  $T_{\text{ad}}(k)$ . The author, in his Ph.D. thesis [Mis13, Theorem 2.2.1] (also [Mis12, Thm. 1]) constructs a map  $T_{\text{ad}}(k)/p(T(k)) \rightarrow \widehat{\mathcal{S}}_\varphi$ . For the action of  $T_{\text{ad}}(k)$  on  $\widehat{\mathcal{S}}_\varphi$  given by this map, he shows that  $\pi_{t \cdot \rho}^{t \cdot [K]} \neq 0 \iff \pi_\rho^{[K]} \neq 0$  for all  $t \in T_{\text{ad}}(t)$ ,  $\rho \in \widehat{\mathcal{S}}_\varphi$ , where  $K$  is a hyperspecial subgroup of  $G(k)$ . Using this result, we have

**Theorem 12.** *Let  $\Pi_\varphi$  be an unramified  $L$ -packet associated to a Langlands parameter  $\varphi$ . Then  $\pi_\rho \in \Pi_\varphi$  is  $\psi$ -generic iff  $\pi_{t \cdot \rho}$  is  $t \cdot \psi$  generic for all  $t \in T_{\text{ad}}(k)$ .*

*Proof.* Given  $\pi_\rho \in \Pi_\varphi$ , let  $K$  be a hyperspecial subgroup such that  $\pi_\rho^{[K]} \neq 0$ . We can write  $K$  as the stabilizer  $G(k)_x$  of some hyperspecial point  $x$  in the Bruhat-Tits building of  $G(k)$ . Without loss of generality we can assume  $x$  to lie in the apartment associated to  $T$ . We have that  $(\text{Ind}_{U(k)}^{G(k)} \psi)^{G(k)_x} \neq 0$  iff there exists  $g \in G$  such that  $\psi|_{g^{-1}G(k)_x g \cap U(k)} \equiv 1$ . Without loss of generality, we can assume that  $g = 1$ . Let  $t \in T_{\text{ad}}(k)$ .

$$\begin{aligned} \text{Hom}_{G(k)}(\pi_\rho, (\text{Ind}_{U(k)}^{G(k)} \psi)) \neq 0 & \quad \text{iff} \quad (\text{Ind}_{U(k)}^{G(k)} \psi)^{G(k)_x} \neq 0 \\ & \quad \text{iff} \quad \psi|_{G(k)_x \cap U(k)} \equiv 1 \\ & \quad \text{iff} \quad t \cdot \psi|_{G(k)_{t \cdot x} \cap U(k)} \equiv 1 \\ & \quad \text{iff} \quad (\text{Ind}_{U(k)}^{G(k)} t \cdot \psi)^{t \cdot [G(k)_x]} \neq 0 \\ & \quad \text{iff} \quad \text{Hom}_{G(k)}(\pi_{t \cdot \rho}, (\text{Ind}_{U(k)}^{G(k)} t \cdot \psi)) \neq 0 \end{aligned}$$

□

*Remark 13.* Note that we do not assume  $\varphi$  to be tempered. However, if the associated  $L$ -packet is not generic, then the above statement could be vacuous.

*Remark 14.* Theorem 12 is a very special case of Conjecture 6'. In [Kal13, Thm. 3.3], Kaletha proves Conjecture 6' for tempered representations in the case when  $G$  is a quasi-split real  $K$ -group or a quasi-split  $p$ -adic classical group (in the sense of Arthur).

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#### REFERENCES

- [Art06] J. Arthur. A note on  $L$ -packets. *Pure Appl. Math. Q.*, 2(1, Special Issue: In honor of John H. Coates. Part 1):199–217, 2006.

- [Ayy13] J. An, J.-K. Yu, and J. Yu. On the dimension datum of a subgroup and its application to isospectral manifolds. *J. Differential Geom.*, 94(1):59–85, 2013.
- [BGA14] M. Borovoi and C. D. González-Avilés. The algebraic fundamental group of a reductive group scheme over an arbitrary base scheme. *Cent. Eur. J. Math.*, 12(4):545–558, 2014.
- [Bor79a] A. Borel. Automorphic  $L$ -functions. In *Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.
- [Bor79b] A. Borel. Automorphic  $L$ -functions. In *Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.
- [DR10] S. DeBacker and M. Reeder. On some generic very cuspidal representations. *Compos. Math.*, 146(4):1029–1055, 2010.
- [GGP12] W.-T. Gan, B. H. Gross, and D. Prasad. Symplectic root numbers, central critical values, and restriction problems in the representation theory of classical groups. *Asterisque*, 2012.
- [Kal13] T. Kaletha. Genericity and contragredience in the local Langlands correspondence. *Algebra Number Theory*, 7(10):2447–2474, 2013.
- [Kan01] R. Kane. *Reflection groups and invariant theory*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 5. Springer-Verlag, New York, 2001.
- [Kar11] D. A. Karpuk. Weil-étale Cohomology over  $p$ -adic Fields. *ArXiv e-prints*, November 2011.
- [Key82] D. Keys. Reducibility of unramified unitary principal series representations of  $p$ -adic groups and class-1 representations. *Math. Ann.*, 260(4):397–402, 1982.
- [Key87] D. Keys.  $L$ -indistinguishability and  $R$ -groups for quasisplit groups: unitary groups in even dimension. *Ann. Sci. École Norm. Sup. (4)*, 20(1):31–64, 1987.
- [KP13] T. Kamran and R. Plymen.  $K$ -theory and the connection index. *Bull. Lond. Math. Soc.*, 45(1):111–119, 2013.
- [Kuo02] W. Kuo. Principal nilpotent orbits and reducible principal series. *Represent. Theory*, 6:127–159 (electronic), 2002.
- [Kuo10] W. Kuo. The Langlands correspondence on the generic irreducible constituents of principal series. *Canad. J. Math.*, 62(1):94–108, 2010.
- [Mil06] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.
- [Mis12] M. Mishra. Structure of the Unramified  $L$ -packet. *ArXiv e-prints*, December 2012.
- [Mis13] M. Mishra. *Structure of the unramified  $L$ -packet*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)—Purdue University.
- [Mis15] M. Mishra. Langlands parameters associated to special maximal parahoric spherical representations. *Proc. Amer. Math. Soc.*, 143(5):1933–1941, December 2015.
- [Ree10] M. Reeder. Torsion automorphisms of simple Lie algebras. *Enseign. Math. (2)*, 56(1-2):3–47, 2010.
- [Ser97] J.-P. Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.
- [Sha90] F. Shahidi. A proof of Langlands’ conjecture on Plancherel measures; complementary series for  $p$ -adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990.

- [Spr79] T. A. Springer. Reductive groups. In *Automorphic forms, representations and  $L$ -functions* (*Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977*), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.
- [Yu] J.-K. Yu. A note on the relative root datum of quasi-split groups (preprint).

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